

*Proceedings of the 27th International Conference on Probabilistic, Combinatorial
and Asymptotic Methods for the Analysis of Algorithms
Kraków, Poland, 4-8 July 2016*

Recent results on permutations without short cycles

Robertas Petuchovas

Vilnius University
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
e-mail: robertas.petuchovas@mif.vu.lt

Abstract. The density, denoted by $\kappa(n, r)$, of permutations having no cycles of length less than $r + 1$ in a symmetric group S_n is explored. New asymptotic formulas for $\kappa(n, r)$ are obtained using the saddle-point method when $5 \leq r < n$ and $n \rightarrow \infty$.

Keywords: symmetric group, long cycles, Buchstab's function, Dickman's function, saddle-point method

The probability $\kappa(n, r)$ that a permutation sampled from the symmetric group S_n uniformly at random has no cycles of length less than $r + 1$, where $1 \leq r < n$ and $n \rightarrow \infty$, is explored. New asymptotic formulas valid in specified regions are obtained using the saddle-point method. One of the results is applied to show that estimate of the total variation distance for permutations can be expressed only through the function $\nu(n, r)$ which is a probability that a permutation sampled from the S_n uniformly at random has no cycles of length greater than r .

To address the problem, we need recollect the following functions. Buchstab's function $\omega(v)$ is defined as a solution to difference-differential equation

$$(v\omega(v))' = \omega(v - 1)$$

for $v > 2$ with the initial condition $\omega(v) = 1/v$ if $1 \leq v \leq 2$. Dickman's function $\varrho(v)$ is the unique continuous solution to the equation

$$v\varrho'(v) + \varrho(v - 1) = 0$$

for $v > 1$ with initial condition $\varrho(v) = 1$ if $0 \leq v \leq 1$.

The interest to the problem begins with the classical example of derangements

$$\kappa(n, 1) = \sum_{j=0}^n \frac{(-1)^j}{j!} = e^{-1} + O\left(\frac{1}{n!}\right)$$

and the trivial case $\kappa(n, r) = 1/n$ if $n/2 \leq r < n$. There was a series of works concerning general asymptotic formulas of the probability $\kappa(n, r)$ the strongest of which are presented here as Proposition 1 and Proposition 2.

Proposition 1 For $1 \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r + \gamma} \omega(n/r) + O\left(\frac{1}{r^2}\right).$$

See [6, Theorem 3].

Proposition 2 Let $u = n/r$. For $1 \leq r \leq n/\log n$,

$$\kappa(n, r) = e^{-H_r} + O\left(\frac{(u/e)^{-u}}{r^2}\right).$$

If $r \geq 3$, we can replace e by 1 in the error term.

See [12, Proposition 2]. Together these propositions provide stronger estimates of $\kappa(n, r)$ than those in [2], [3], [4]. New results are the following theorems:

Theorem 1 For $\sqrt{n \log n} \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r + \gamma} \omega(n/r) + O\left(\frac{\varrho(n/r)}{r^2}\right).$$

Proof. The result is a corollary of Theorem 1 in [7]. It is obtained from the probability generating function using saddle-point method, the technique is elaborated in [11].

Theorem 2 For $(\log n)^4 \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r} + O\left(\frac{\varrho(n/r)}{r}\right).$$

Proof. The saddle-point method is applied to the Cauchy's integral representation of $\kappa(n, r)$, as in the proof of Theorem 1. However, there are some other technical difficulties one must to overcome.

Theorem 3 For $5 \leq r < n$, we have

$$\kappa(n, r) = e^{-H_r} + O\left(\frac{\nu(n, r)}{r}\right).$$

Proof. Quite the same technique to that used in the proof of Theorem 2 is employed, just a different approximation of the saddle point is taken and Corollary 5 of [8] is applied.

Theorem 1 and Theorem 2 (see also Corollary 2.3 in [5]) improve on Proposition 1 and Proposition 2. Theorem 3 is of separate interest; as we see, it can be useful in formulas where both probabilities $\kappa(n, r)$ and $\nu(n, r)$ are involved. Here is an example.

Let $k_j(\sigma)$ equal the number of cycles of length j in a permutation $\sigma \in S_n$, $\bar{k}(\sigma) = (k_1(\sigma), k_2(\sigma), \dots, k_n(\sigma))$, and $\bar{Z} = (Z_1, Z_2, \dots, Z_n)$, where Z_j are Poisson random variables such that $E Z_j = 1/j$, $j \in \mathbb{N}$. Thus, if $5 \leq r < n$, we have (see Lemma 3.1 on p. 69 of [1])

$$\begin{aligned} d_{TV}(n, r) &= \sup_{V \subseteq \mathbb{Z}_+^r} \left| \frac{\#\{\sigma : \bar{k}(\sigma) \in V\}}{n!} - \Pr(\bar{Z} \in V) \right| \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \nu(m, r) |\kappa(n-m, r) - e^{-H_r}| \\ &= \frac{e^{-H_r}}{2} \sum_{m=n-r}^{\infty} \nu(m, r) + \frac{1}{2} \nu(n, r) + O\left(\frac{1}{r} \sum_{m=0}^{n-r-1} \nu(m, r) \nu(n-m, r)\right). \end{aligned}$$

Consequently, only results on the probability $\nu(n, r)$ are needed attempting to improve on the order of notable estimate for $d_{TV}(n, r)$ in [2].

References

- [1] R. Arratia, A.D. Barbour, and S. Tavaré, *Logarithmic Combinatorial Structures: A Probabilistic Approach*, EMS Publishing House, Zürich, 2003.
- [2] R. Arratia and S. Tavaré, The cycle structure of random permutations, *Ann. Probab.*, 1992, **20**, 3, 1567-1591.
- [3] E.A. Bender, A. Mashatan, D. Panario, and L.B. Richmond, Asymptotics of combinatorial structures with large smallest component, *J. Comb. Th.*, Ser. A, 2004, **107**, 117–125.
- [4] A. Granville, Cycle lengths in a permutation are typically Poisson, *Electronic J. Comb.*, 2006, **13**, #R107.
- [5] A. Hildebrand and G. Tenenbaum, Integers without large prime factors, *J. Théorie des Nombres de Bordeaux*, 1993, **5**, 411–484.
- [6] E. Manstavičius, On permutations missing short cycles, *Lietuvos matem. rink.*, spec. issue, 2002, **42**, 1–6.
- [7] E. Manstavičius and R. Petuchovas, Local probabilities and total variation distance for random permutations, *The Ramanujan J.*, 2016, DOI 10.1007/s11139-016-9786-0.
- [8] E. Manstavičius and R. Petuchovas, *Local probabilities for random permutations without long cycles*, *Electron. J. Combin.*, 2016, **23(1)**, #P1.58.
- [9] E. Manstavičius and R. Petuchovas, Permutations without long or short cycles, *Electronic Notes in Discrete Mathematics*, 2015, **49**, 153-158.
- [10] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, 1995.

- [11] G. Tenenbaum, Crible d'Ératosthène et modèle de Kubilius. In: *Number Theory in Progress, Proc. Conf. in Honor of Andrzej Schinzel*, Zakopane, Poland, 1997. K. Gyory, H. Iwaniec, J. Urbanowicz (Eds.), Walter de Gruyter, Berlin, New York, 1999, 1099–1129.
- [12] A. Weingartner, On the degrees of polynomial divisors over finite fields, arXiv:1507.01920.